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IMPROVING THE AERODYNAMIC PERFORMANCE OF SMALL-ASPECT-RATIO WINGS AT HYPERSONIC SPEEDS†

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Present approaches to improving the aerodynamic performance of lifting surfaces at hypersonic speeds rely on the elementary Newton formula for the pressure at a surface [1, 2]. Aerodynamic shapes of fairly high supersonic and hypersonic performance have also been developed as waveriders—bodies formed by stream surfaces of inviscid flows behind the shock waves of simple configurations, for which exact solutions exist [3–6].

In this paper the variational problem for the shape of a thin lifting surface of high hypersonic performance is formulated by using analytical expressions obtained in [7, 8] for the flow around a small-aspect-ratio wing at an angle of attack in the next approximation to the Newtonian theory. The solutions of this problem define the planform of the lower surface of the wing and its leading edge, ensuring an appreciable increase in aerodynamic performance compared with the maximum of Newtonian theory, for a given lift force or payload.

1. WE WILL consider the flow in the thin compressed layer between the forward shock and windward surface of a wing, in a hypersonic flow of air moving past the wing without slip at an angle of attack α, in a coordinate system $x°y°z°$ (Fig. 1). The wing thickness, measured from the base plane $y° = 0$, is assumed to be small. As the forward shock layer is strongly compressed, its surface is also close to the plane $y° = 0$ and the small parameter of the thin shock layer method, equal to the density ratio at the shock wave, becomes

$$\varepsilon=\frac{\varkappa-1}{\varkappa+1}+\frac{2}{(\varkappa+1)M_\infty^2\sin^2\alpha} \tag{1.1}$$

where $\varkappa$ is the adiabatic exponent and M_∞ is the Mach number of the oncoming stream. The passage to the limit $\varepsilon\to 0$ ($\varkappa\to 1$, $M_\infty\to\infty$) is subject to the condition

$$R=\varepsilon\varkappa M_\infty^2\sin^2\alpha=O(1) \tag{1.2}$$

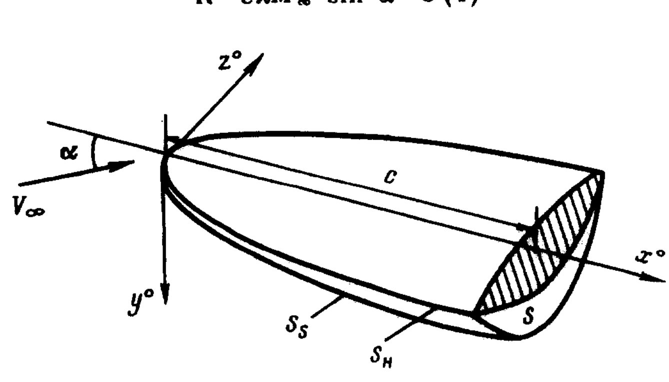

FIG. 1.

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We shall assume that the aspect ratio is of the same order of magnitude as the Mach angle $\varepsilon^{1/2}\operatorname{tg}\alpha$ in the disturbed flow [7, 8]—this is the most interesting case of three-dimensional flow around a wing.

The first-order dimensionless coordinates in the shock layer, as $\varepsilon\to 0$ are

$$x=\frac{x^0}{c},\quad y=\frac{y^0}{c\varepsilon\operatorname{tg}\alpha},\quad z=\frac{z^0}{c\varepsilon^{1/2}\operatorname{tg}\alpha} \tag{1.3}$$

where c is a unit of length.

The asymptotic expansions (small ε) of the velocity components $\mathbf{V}^\circ$ (u°, v°, w°) along the x°, y°, z° axes, the pressure p° and density ρ° in the shock layer are

$$\begin{gathered}u^\circ/V_\infty=\cos\alpha+\varepsilon u(x,y,z)\sin\alpha\operatorname{tg}\alpha+O(\varepsilon^2)\\ v^\circ/V_\infty=\varepsilon v(x,y,z)\sin\alpha+O(\varepsilon^2)\\ w^\circ/V_\infty=\varepsilon^{1/2}w(x,y,z)\sin\alpha+O(\varepsilon^{3/2})\\ (p^\circ-p_\infty)/\rho_\infty V_\infty^2=[1+\varepsilon p(x,y,z)]\sin^2\alpha+O(\varepsilon^2)\\ \rho^\circ/\rho_\infty=\varepsilon^{-1}+\rho(x,y,z)+O(\varepsilon)\end{gathered} \tag{1.4}$$

Substituting (1.4) into the equations of gas-dynamics, we obtain a system of equations of the next approximation to the Newtonian one

$$\begin{gathered}Du=Dw=0,\quad Dv=-p_y,\quad v_y+w_z=0\\ D=\partial/\partial x+v\partial/\partial y+w\partial/\partial z\end{gathered} \tag{1.5}$$

As boundary conditions for (1.5) we assume that the wing surface $y=b(x,z)$ is impermeable and we impose certain restrictions on the shock $y=s(x,z)$ of which we present only the two that will be needed later:

$$u^s(x,z)=-s_x(x,z),\quad w^s(x,z)=-s_z(x,z) \tag{1.6}$$

It follows from Eqs (1.5) that the transverse component of the velocity w is conserved along the disturbed flow streamlines and may be taken as one of the two independent stream functions:

$$\psi=w,\quad \chi=z-\psi x \tag{1.7}$$

Suppose that the streamline passing through a point (x, y, z) in the shock layer crosses the shock wave at a point $[\xi, s(\xi,\zeta), \zeta]$. The equation of the streamline in terms of ξ and ζ is obtained from (1.6), (1.7):

$$\zeta=z+(x-\xi)s_z(\xi,\zeta) \tag{1.8}$$

In the case of an attached forward shock, the streamlines running along the wing surface cross the shock at points on the leading edge $z=z_e(x)$. Hence, the abscissae $\xi^e(x,z)$ of these points satisfy the equation $\zeta(x,\xi^e,z)=z_e(\xi^e)$.

The differentials dy and $d\xi$ satisfy the following relationship for constant x, z [8]: $dy=\zeta_z(x,\xi,z)d\xi$. Integrating, we obtain a relationship between the shape of the shock wave and the shape of the body [7, 8]:

$$s(x,z)=b(x,z)+\int_{\xi^e(x,z)}^{x}\zeta_z(x,\xi,z)\,d\xi \tag{1.9}$$

This equation must be solved simultaneously with (1.8).

2. Let us apply the laws of conservation of mass and momentum to the volume of gas enclosed by the shock surface S_S, the windward side of the body S_H and the rear side of the shock layer S, i.e. its section by the plane $x=1$ (Fig. 1). Since the viscosity of air is negligible and the pressure on the windward side of the wing and its bottom surface may be taken to be equal to zero, allowance for the impermeability of S_H gives

$$\mathbf{F} = -\iint_S [p^\circ \mathbf{i} + \rho^\circ (\mathbf{V}^\circ - \mathbf{V}_\infty) u^\circ]\, dS - p_\infty \iint_{S_s} \mathbf{n}\, dS$$

where $\mathbf{n}$ is the unit normal to the shock surface. Projection of F onto the y° and x° axes gives the required components of the aerodynamic force in the attached coordinate system:

$$Y = \iint_S \rho^\circ (v^\circ - v_\infty) u^\circ dy^\circ dz^\circ + p_\infty S_\Gamma{}^\circ \tag{2.1}$$

$$X = \iint_S [p^\circ - p_\infty + \rho^\circ (u^\circ - u_\infty) u^\circ] dy^\circ dz^\circ + p_\infty S_B{}^\circ \tag{2.2}$$

where $S_\Gamma{}^\circ$, $S_B{}^\circ$ are the areas of the projections of the wing onto the $x^\circ z^\circ$ and $y^\circ z^\circ$ planes. By (1.3), the area of the rear side of the shock layer is

$$S^\circ = 2\varepsilon^{1/2} c^2 \sigma \operatorname{tg}^2 \alpha, \quad \sigma = \iint_S dy dz$$

Substituting (1.3) and (1.4) into (2.1) and (2.2) and using (1.2), we obtain expansions of the components of the force in powers of ε:

$$\frac{Y}{\rho_\infty V_\infty{}^2 c^2} = 2\varepsilon^{1/2} \sigma (1 + \varepsilon P) \sin^2 \alpha \operatorname{tg} \alpha + O(\varepsilon^{5/2}) \tag{2.3}$$

$$P = \frac{\sigma_\Gamma}{R\sigma} + \frac{1}{\sigma} \iint_S (v + \rho + u \operatorname{tg}^2 \alpha)\, dy dz$$

$$\sigma_\Gamma = S_\Gamma{}^\circ / (2\varepsilon^{1/2} c^2 \operatorname{tg} \alpha)$$

$$\frac{X}{\rho_\infty V_\infty{}^2 c^2} = -2\varepsilon^{3/2} \sigma (1 - Q) \sin^2 \alpha \operatorname{tg}^2 \alpha + O(\varepsilon^{5/2}) \tag{2.4}$$

$$Q = -\frac{1}{\sigma} \iint_S u\, dy dz$$

Hence the aerodynamic performance is (in the velocity system of coordinates)

$$K = \operatorname{ctg} \alpha + \frac{\varepsilon}{\sin \alpha \cos \alpha} (1 - Q) + O(\varepsilon^2) \tag{2.5}$$

The principal term here is the Newtonian quantity $\operatorname{ctg}\alpha$. To determine the correction to first order in ε, the only function we need to know is u.

Let us transform the integrals in the formula for Q. We will first change the variables from y, z to ξ, z [$\beta = z_e(1)$]:

$$\int_0^\beta \int_{b(1,z)}^{s(1,z)} u\, dy\, dz = \int_0^\beta \int_{\xi^e(1,z)}^{1} u \zeta_z (1, \xi, z)\, d\xi\, dz$$

By the first equation of (1.5), the longitudinal component of the velocity is conserved along streamlines in the shock layer. With due attention to the boundary condition (1.6), we see that along the streamlines that cross the shock wave at (ξ, ζ) we have $u = s_x(\xi, \zeta)$. We need another change of variables, from ξ, z to ξ, ζ. The Jacobian of the transformation is

$$\frac{\partial(\xi, z)}{\partial(\xi, \zeta)} = \left.\frac{\partial z(x, \xi, \zeta)}{\partial \zeta}\right|_{x=1} = \frac{1}{\zeta_z(1, \xi, z)}$$

so that the derivative ζ_z in the integrand cancels out. The boundary of the integration domain with respect to ξ, z consists of three parts: the leading edge $\xi = \xi^e(1, z)$, a segment of the axis $z = 0$ and the rear side $\xi = 1$. In terms of ζ,ξ these three parts are described by the equations $\xi = x_e(\zeta)$, $\zeta = 0$ and $\xi = 1$, so that the new limits of integration are $x_e(\zeta)$ and 1 (with respect to ξ), 0 and $\beta = z_e(1)$ (with respect to ζ). This change of variables is legitimate provided the transformation $\zeta = \zeta(x, \xi, z)$ is one-to-one; in other words, the curves ζ = const should not intersect one another or hit the leading edge twice. In addition, the curves $\zeta > 0$ should not cut the plane $z = 0$. Violation of these conditions would mean that the streamlines intersected at points of entry to the compressed layer or in the plane of symmetry—rendering the situation physically meaningless. In the inverse flow problem (considered in this paper) these conditions must be interpreted as constraints on the distribution of the transverse inclinations $s_z(x, z)$ of the shock layer.

The integral Q, written in terms of ξ, ζ becomes

$$Q = \frac{1}{\sigma}\int_0^{\beta}\int_{x_e(\zeta)}^{1} s_x(\xi, \zeta)\, d\xi d\zeta$$

Since ζ is fixed in the inner integral, $s_x(\xi, \zeta)$ is the derivative $\partial s(\xi, \zeta)/\partial\xi$. Integrating with respect to ξ, changing the notation ζ to z and similarly transforming the integral σ, we finally obtain $\sigma = \sigma_\Gamma$ and

$$Q = \int_0^{\beta} \{s(1, z) - s[x_e(z), z]\}\, dz \Big/ \int_0^{\beta} [1 - x_e(z)]\, dz \tag{2.6}$$

The one-dimensional integrals in this formula have a simple geometrical meaning. The integral in the denominator is the area of the wing in plan, the integral in the numerator is the area between the rear side of the shock layer $x = 1$ and the projection of the leading edge of the wing onto that plane. To obtain possibly much superior performance in the next approximation to the Newtonian one, we must require that $A < 0$, that is, the projection of the edge must lie beneath the section of the shock layer. This may be guaranteed by "bending" the forward part of the wing downward, which has the effect, first of all, of reducing wave drag.

3. We will distinguish partial and total optimization: partial optimization means increasing the aerodynamic performance K (2.5) by minimizing the functional (2.6) for a given shape of the forward shock wave $s(x, z)$ and total optimization involves determining the shape of the shock wave for which partial optimization yields the largest value of K.

It can be shown that both partial and total optimization problems have a non-trivial solution only under certain additional constraints, e.g. on the magnitude of the lift, the geometrical parameters of the lifting body, and so on. Here we will consider constraints of the isoperimetric type, i.e. expressed solely in terms of integrals of the unknown function.

If the shock is represented by a grid function and the problem of total optimization is treated by methods for minimizing functions of several variables, high accuracy can be achieved if the grid is fine enough. However, this approach requires a prohibitive amount of computer time on presently available hardware. We will therefore use another method here, which also yields an approximate solution of the problem of total optimization: we will attempt to optimize the solution for a shock wave whose shape is given by a function $s(x, z; P_1, \ldots, P_N)$ depending on several parameters, and then vary the parameters P_n over a certain set to get the required solution. We define s as follows:

$$s(x, z) = a(x) z^n + a_0(1 - x), \quad a(x) = k \ln(\delta + x) \tag{3.1}$$

where $n > 0$, k, $\delta > 0$ and a_0 are parameters defining the shape of the shock.

The partial optimization problem, in its most general form, is

$$Q_{\text{opt}} = \min Q[z_e(x)], \quad z_e(x) \in Z, \quad s(x, z) \in \Sigma \tag{3.2}$$

$$G[z_e(x)] = G_0 \tag{3.3}$$

Here $z_e(x)$ is the unknown function, which describes the planform of the leading edge and, once the shock is given, uniquely defines the wing surface, Z is a set (class) of functions $z_e(x)$ that satisfy (3.3) and certain other conditions (smoothness, symmetry and so on), Σ is the set of function $s_e(x)$ defined by (3.1), G is a functional in the isoperimetric condition (3.3) and G_0 is a given value of G. The problem of total optimization will be solved by confining the search to a discrete subset Σ' of Σ.

A necessary and sufficient condition for (3.2) to hold, subject to constraints (3.3), is that the first variation δQ must vanish and the second variation $\delta^2 Q$ must be positive over the set Z:

$$\delta Q=0, \quad \delta^2 Q>0, \quad z_e(x)\in Z \tag{3.4}$$

These conditions are only satisfied for $z_e(x)\in Z$, implying the condition

$$\delta G[z_e(x)]=0 \tag{3.5}$$

which is obtained by varying (3.4). Thus, conditions (3.4) have to be observed not for any variation $\delta z_e(x)$ (as in unconstrained optimization) but only for variations $\delta z_e(x)$ that satisfy (3.5).

The first and second variation of the functional (2.6) can be written as

$$\delta Q=\frac{1}{\sigma_\Gamma}[\delta\sigma_c-Q\delta\sigma_\Gamma], \quad \delta^2 Q=\frac{1}{\sigma_\Gamma}[\delta^2\sigma_e-2\delta\sigma\delta Q]$$

$$\delta\sigma_e=\int_0^1 s_x[x, z_e(x)]\,\delta z_e(x)\,dx, \quad \delta\sigma_\Gamma=\int_0^1 \delta z_e(x)\,dx \tag{3.6}$$

$$\delta^2\sigma_e=\int_0^1 s_{xz}[x, z_e(x)]\,[\delta z_e(x)]^2\,dx$$

Let $z_e(x)$ be a function satisfying (3.3) whose variation δz_e makes (3.5) an identity (in short, an "admissible" function). If also $\delta Q=0$, then $z_e(x)$ is an extremal curve and, by (3.6), the sufficient conditions for Q to have a minimum become

$$\int_0^1 \{s_x[x, z_e(x)]-Q\}\,\delta z_e(x)\,dx=0, \quad \delta G[z_e(x)]=0 \tag{3.7}$$

$$\int_0^1 s_{xz}[x, z_e(x)]\,[\delta z_e(x)]^2\,dx>0 \tag{3.8}$$

The last condition is automatically satisfied if $s_{xz}(x, z)>0$ throughout the region of interest. We will now write down (3.7) and (3.8) for a power-shaped shock (3.1). Substituting (3.1) into (3.7), we obtain

$$\int_0^1 [a'(x)z_e^n(x)-a_0-Q]\,\delta z_e(x)\,dx=0 \tag{3.9}$$

Since $s_{xz}(x, z)=na'(x)z^{n-1}=nkz^{n-1}/(\delta+x)$, condition (3.8) implies the following sufficient condition for maximum aerodynamic performance if $\delta>0$: $k>0$ or $\operatorname{sign} a'(x)=1$.

4. We will now solve the partial optimization problem for some specific constraints (3.3). We will first consider enhancement of aerodynamic performance for a given lift. By (2.3), specification of lift is equivalent to specification of the wing area, and condition (3.3) becomes

$$\sigma_\Gamma=\sigma=\int_0^\beta [1-x_e(z)]\,dz=\int_0^1 z_e(x)\,dx=\sigma_0 \tag{4.1}$$

Using (3.9) and (4.1), we deduce from (3.7) that

$$\int_0^1 \{a'(x)z_e^n(x)-a_0-Q\}\,\delta z_e(x)\,dx=0, \quad \int_0^1 \delta z_e(x)\,dx=0 \tag{4.2}$$

It can be proved that if both functionals (4.2) vanish simultaneously, the coefficients of $\delta z_e(x)$ in both will be proportional (this is the essence of the method of Lagrange multipliers). Thus, we obtain

$$a'(x)z_e^n(x)-a_0-Q=\lambda=\mathrm{const}$$

when it follows that

$$z_e(x) = \left[\frac{Q + a_0 + \lambda}{a'(x)}\right]^{1/n} \tag{4.3}$$

Substituting this equality into the formula for Q for a power-shaped shock,

$$Q = \frac{1}{\sigma_\Gamma}\int_0^1 a'(x)\,\frac{z_e^{n+1}(x)}{n+1}\,dx - a_0 \tag{4.4}$$

and using (4.1), we find Q as a function of λ:

$$Q = \lambda/n - a_0 \tag{4.5}$$

Substituting (4.3) and (4.5) into (4.1), we obtain

$$\lambda = \frac{n}{n+1}\,\sigma_0^n\left[\int_0^1 |a'(x)|^{-1/n}\,dx\right]^{-n} \operatorname{sign} a'(x) \tag{4.6}$$

Thus the function $z_e(x)$ is fully defined and, performing the integration in (1.9) by the methods, say, in [9], we can determine the optimum wing surface.

Another constraint that is frequently met with in practice is a given volume of the lifting body. In shock-wave theory, when the pressure on the windward side of the body is negligibly small, such a constraint is, strictly speaking, not legitimate. It is more correct to limit the volume of the lower part of the body, measuring it from the local level of the wing and so on. In the general case, we may assume that some quantity

$$V = \iint_{\sigma_\Gamma} \{b(x, z) - k_V b[x, z_e(x)]\}\,dx\,dz = V_0 \tag{4.7}$$

is given, where $0 \leqslant k_V \leqslant 1$ is a modified parameter characterizing the choice of the reference level. We note that $b[x, z_e(x)] = s[x, z_e(x)]$ because the forward shock is attached. In addition, the function $b(x, z)$ may be isolated from formula (1.9). We may assume without loss of generality that $k_V = (n+1)^{-1}$—this choice greatly simplifies the final results. Then

$$V = \frac{na_0}{n+1}\int_0^1 (1 - x)\,z_e(x)\,dx - \int_0^1 \int_0^{z_e(x)} \int_{\xi^e(x,z)}^{x} \zeta_z(x, \xi, z)\,d\xi\,dz\,dx$$

Dealing with the multiple integral as in the derivation of formula (2.6), we rewrite (4.7) in the form

$$\left(\frac{na_0}{n+1} - 1\right)\int_0^1 (1 - x)\,z_e(x)\,dx = V_0 \tag{4.8}$$

Thus, if the shock is (3.1), the system of equations (3.8), (3.9) takes the form

$$\int_0^1 \{a'(x)\,z_e^n(x) - a_0 - Q\}\,\delta z_e(x)\,dx = 0$$

$$\int_0^1 (1 - x)\,\delta z_e(x)\,dx = 0$$

whence we obtain $a'(x)z_e^n(x) - a_0 - Q = \lambda(1 - x)$, $\lambda = \text{const}$,

$$z_e(x) = \left[\frac{Q + a_0 + \lambda(1 - x)}{a'(x)}\right]^{1/n} \tag{4.9}$$

By (4.9) and (4.4),

$$Q = \frac{\lambda}{n}\,\frac{V_0}{\sigma_\Gamma} - a_0$$

Substituting these expressions into (4.8) and using (4.1), we can then use numerical techniques to solve the resulting equation for σ_Γ and thus determine $z_e(x)$.

With regard to total optimization, we draw the reader's attention to the following point.

Formula (4.4), which is independent of the specific form of the constraint (3.3), makes Q a linear function of the coefficient k (and if $a_0 = 0$ it is even directly proportional to the latter). Hence, if $k<0$, then Q will be smaller, and K will therefore be greater than if $k>0$. However, by (3.1) and (3.10), the solutions (4.9) produce lifting bodies of maximum performance (minimum Q) only when $k>0$. If $k<0$ these solutions describe bodies of minimum performance (maximum Q). There is no contradiction, since (4.3) and (4.9) are solutions of the partial optimization problem for a shock of given shape, while $k<0$ and $k>0$ correspond, of course, to different shocks. Nevertheless, a solution to the total optimization problem must be sought in the region of negative k values.

The fact that for a fixed shock shape and negative k the extremum of the functional (2.6) determines the maximum Q gives grounds for the assumption that Q may be made as large as desired in magnitude by a suitable choice of the edge $z_e(x)$; in other words, one can make Q tend to $-\infty$. Indeed, there are examples (when $n = 2$—even analytical examples) of sequences of functions $z_e(x)$ for which $Q \to -\infty$ ($K \to \infty$), while at the same time $\sigma_\Gamma = \text{const}$. However, the function $z_e(x)$ thus obtained is not monotone (the inverse function $x_e(z)$ is not well defined)—a situation implying that the shape of the leading edge of the lifting body is inadmissible. The important point here is that the inverse problem (1.8), (1.9) is unsolvable, since the function $\xi^e(x, z)$ is multivalued; such conditions are physically meaningless, as they mean that the streamlines intersect one another on the wing.

Thus, the total optimization problem is extremely complicated, since, besides a constraint of the type (3.3), one must also require that the inverse problems should be solvable and that the shape of the forward edge must be geometrically admissible.

5. The direct approach to total optimization consists in selecting a suitable function $s(x, z)$ from some finite subset Σ' of the set Σ of admissible functions (see Sec. 3); for each function $s(x, z)$, the partial optimization problem is also solved by selecting the independent parameters that define the shape of the leading edge. Considerable attention has been devoted to the large class of lifting bodies in which the lateral part of the edge is described by a cubic parabola with axis emanating from a point $z_0 \geqslant 0$, inclined at an angle θ to the x axis (Fig. 2):

$$z_e(x) = \begin{cases} z_0 + x \operatorname{tg}\theta + \Delta z_e(x), & 0 \leqslant x \leqslant x_1 \\ z_1, & x_1 \leqslant x \leqslant 1 \end{cases}$$

$$\Delta z_e \cos\theta = \Phi(x/\cos\theta + \Delta z_e \sin\theta)$$

$$\Phi(\eta) = A\eta(\eta - l)(\eta - L), \quad L = \sqrt{x_1^2 + (z_1 - z_0)^2}, \quad \operatorname{tg}\theta = \frac{z_1 - z_0}{x_1}$$

Here l is determined from conditions (3.3), z_1 is fixed and A, z_0 and x_1 are free parameters.

Solution of the optimization problem for a given lift produced the solution $Q_{\min} = -0.572$ for modified parameter values $\sigma_0 = 1.5$; $z_1 = 1.92$; $z_0 = 0.171$; $x_1 = 1$; $A = -0.5$; $n = 2$; $k = -0.5$;

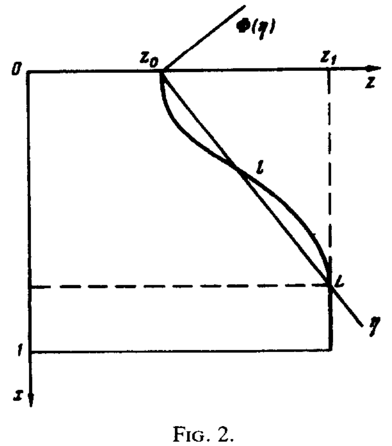

FIG. 2.

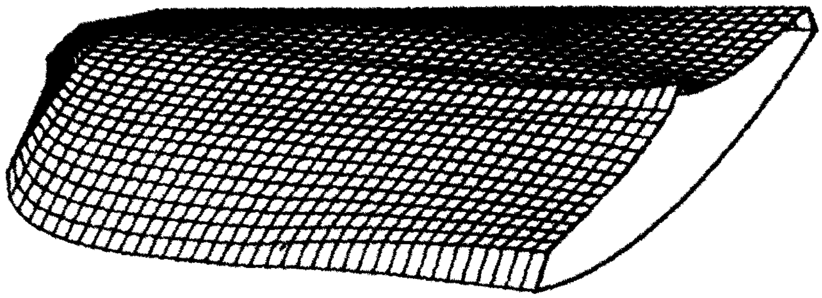

FIG. 3.

$\delta = 0.4$ (Fig. 3). Optimization for a given volume gives $Q_{\min} = -0.592$ for the following modified parameter values: $V_0 = 0.75$; $z_1 = 1.92$; $z_0 = 0.240$; $x_1 = 1$; $A = -0.34$; $n = 2$; $k = -0.5$; $\delta = 0.4$. In both cases, $a_0 = 0$.

The following points are worthy of note:

1. The coefficient k cannot be reduced to less than -0.5, since below this value the inverse problem is no longer solvable.

2. For every k there is a well-defined value of δ at which Q becomes a minimum. In particular, for $k = -0.5$ we obtain $\delta = 0.4$.

3. Despite the fact that for a fixed function $\dot{z}_e(x)$ the integral (4.4) decreases as n increases, the limiting value of the coefficient $k < 0$ for which a solution of the inverse flow problem still exists increases rapidly (i.e. decreases in absolute value), so that the optimum Q is obtained at $n = 2$ (parabolic shock).

The windward surface of the body in both cases considered is blunted ($z_0 > 0$). In addition, in

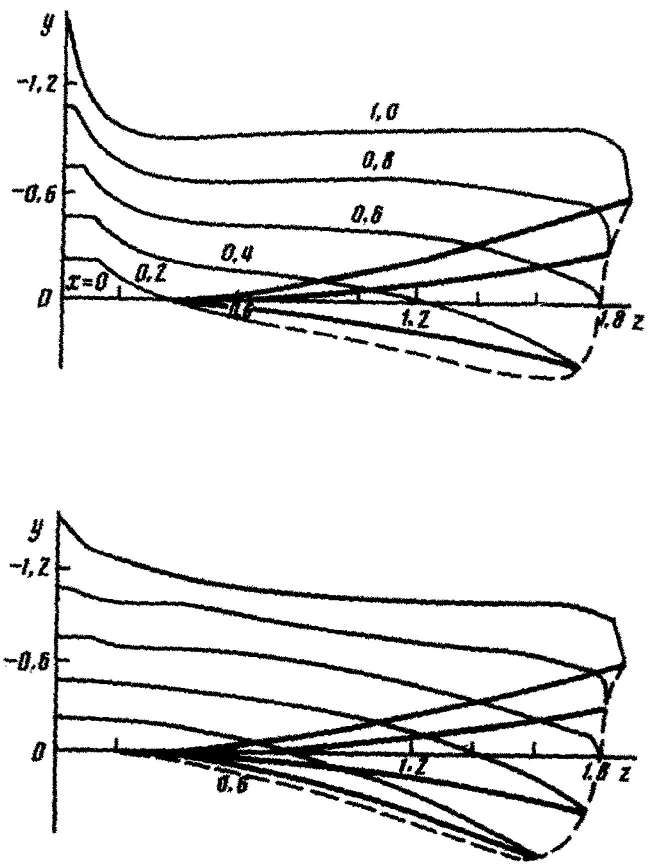

FIG. 4.

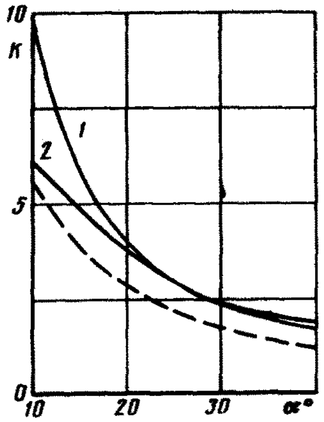

FIG. 5.

order to eliminate the possibility of the edge being cut twice by the surface streamlines, it was necessary to subject the edge to a small parabolic deformation in a narrow strip at the rear. In a flight regime with $\alpha = 25°$ and $\varepsilon = 0.3$ the values of Q determined here guarantee fairly high hypersonic aerodynamic performance, $K = 3.3$, which is superior to that of a plate with the same flight parameters by a factor of 1.5.

Figure 3 illustrates the optimal shapes of the lower surfaces of wings (with superimposed computational grid) and the rear side of the shock layer for the two cases considered. Transverse sections $x =$ const of wings (thin curves) and attached shock are shown in Fig. 4. It is obvious that good results are achieved for a wing with concave lower surface and relatively small blunted section, by downward bending not of the entire surface but of its peripheral part only, adjoining the leading edge (the projection of this part is represented by the dashed curve); even this produces an appreciable effect. To realize this procedure, one can set $a_0 > 0$ in (3.1). The wing then has the same planform, but the magnitude Q receives a negative increment $-a_0$. However, for an optimum wing, even at $a_0 = 0$ the angle of inclination of the velocity vector of the oncoming stream at the lower surface in the longitudinal direction is less than the angle of attack (particularly in the rear), so that flow with the formation of a compressed layer is possible only at very small a_0 values.

If allowance is also made for viscosity, assuming, for simplicity, that the local coefficient of friction c_f is fairly small and constant over the wing area, the aerodynamic performance (2.5) is reduced by an amount $c_f/\sin^4\alpha$. Consequently, a characteristic feature of the flow regime considered, with finite angle of attack, is that the force of friction does not change the shape of the optimal wing and merely reduces the optimum performance; moreover, the effect of friction falls rapidly as α increases (Fig. 5). Curve 1 in the figure corresponds to $c_f = 0$, curve 2 to $c_f = 3 \times 10^{-3}$ and the dashed line to the Newtonian value $K = \operatorname{ctg}\alpha$.

Viscosity may be responsible not only for drag but also for changes of pressure due to interaction of a hypersonic boundary layer with an external inviscid flow. However, the variational problems obtained in this formulation are so complicated—even if one uses the simplest tangent-wedge formula for the pressure—that one can consider only optimization for a given wing planform (see, e.g. [10]). Nevertheless, even these results indicate that, as determined here, optimum wings frequently have a downward bent leading edge.

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THE EFFECT OF MERIDIONAL ELECTRIC VORTEX FLOW ON THE AZIMUTHAL ROTATION OF A FLUID†

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New exact solutions of the Navier–Stokes equations are obtained for spiral axisymmetric flow of a conducting fluid in bounded and unbounded regions. Attention is devoted to the influence of the poloidal component of the velocity field, generated by the meridional electric vortex flow, on the toroidal component due to the rotating boundaries of the region. A two-parameter family of self-similar solutions obtained by numerical integration of a system of non-linear ordinary differential equations is investigated. It is shown, considering twisted flow around a cylinder in an unbounded region and differential rotation between coaxial cylinders, that boundary layer regimes of meridional flow induce a boundary layer structure in the azimuthal rotation of the fluid.

Spiral vortex structures in fluids are of interest in connection with phenomena observed when magnetic fields are excited by moving conducting media (MHD-dynamos), in the formation of large-scale atmospheric eddies, the phenomenon of reverse energy cascade in turbulence, etc. In magnetohydrodynamics, three-dimensional vortex flows and magnetic fields are conveniently split (depending on the phenomenon under consideration) into mutually interacting toroidal and poloidal components [1]. Electric vortex (EV) flows, which are created by the interaction of a non-uniform electric current and an intrinsic magnetic field, are of particular interest in MHD. When EV flows are investigated in axially symmetric situations, one can find self-similar solutions of the MHD equations. In that case, however, only poloidal flows are possible. Toroidal flows, set up in the absence of external magnetic fields by azimuthal currents only, are not observed, since in an axially symmetric situation the φ-component of the electric field may arise neither from the action of external sources nor by induction from the motion of the fluid [2]. To organize a spiral structure, the

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